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Probabilistic Fuzzy Sets and a Related Operator Algebra

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The rules of union and intersection of probabilistic fuzzy sets guided us to construct a related operator algebra. In a Hilbert space, where each fuzzy set is represented by an orthonormal vector, the union and the intersection operators generate a well-defined algebra with a unique representation.

KEY WORDS: quantum physics; Hilbert space; fuzzy sets; random sets; q-oscillator; operator algebra.
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1. INTRODUCTION

Fuzzy sets since their introduction (Zadeh, 1965, 1978; Zimmermann, 1984) have received a great deal of interest. For an ordinary set, a given object either belongs or does not belong to the set, whereas for a fuzzy set the degree of membership of an object is given by the value of the membership function for the object. Hence, one can say that there is an uncertainty associated with the membership of an element to a given fuzzy set. This is reminiscent of the uncertainty principle in quantum physics where two observables such as momentum and position along a given direction can not be measured simultaneously with perfect precision. In quantum theory, physical observables are described by (noncommuting) Hermitian linear operators acting on a complex Hilbert space. This quantization process forms the basis of quantum theory. Well-known examples of algebras defined by relations satisfied by such operators are

$$a\,a^* - a^*a = 1 \quad \text{Bosonic algebra}$$

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$$a\,a^* + a^*a = 1; a^2 = 0 \quad \text{Fermionic algebra}$$

$$J_1 J_2 - J_2 J_1 = i\,J_3;\ \text{cyclic permutations Angular momentum algebra.}$$

The first and the last of these systems form a Lie algebra whereas the second one forms a Lie super-algebra. With the introduction of q-oscillators (Arik and Coon, 1976; Biedenharn, 1989; Bonatsos *et al.*, 1991; Chakrabarti and Jagannathan, 1991; Chaichian, 1990; Chaichian *et al.*, 1991; Daskaloyannis, 1991; Fairlie and Zachos, 1991; Hayashi, 1990; Jannussis *et al.*, 1991; Macfarlane, 1989) and quantum groups (Drinfeld, 1986; Fadeev *et al.*, 1989; Jimbo, 1986; Woronowicz, 1987), algebras which do not fall into these categories, but in the limit $q \to 1$ of the deformation parameter reduce to Lie algebra or Lie super-algebra were considered.

This paper may be considered as a first step in establishing a relationship between fuzzy sets and quantum physics. We introduce an algebra related to probabilistic fuzzy sets. We will take the membership functions of the different elements to be represented by commuting Hermitian operators. This is the simplest possible choice for quantization and most probably a consistent and useful quantization of fuzzy sets will involve noncommuting membership functions for different elements. However, as we will show, even our simple choice yields a highly nontrivial operator algebra for union and intersection operators.

This paper is partly motivated by the algebra of the quantum matrix group $SU_q(2)$ (Drinfeld, 1986; Faddeev *et al.*, 1989; Jimbo, 1986; Woronowicz, 1987). The relations below form the algebra

$$a^*a + q^{-1}\,b\,b^* = 1 \tag{1}$$

$$a\,a^* + b\,b^* = 1 \tag{2}$$

$$b\,b^* = b^*b \tag{3}$$

$$a\,b = \sqrt{q}\,b\,a \tag{4}$$

$$a\,b^* = \sqrt{q}\,b^*a \tag{5}$$

where $0 < q \leq 1$ and the $SU_q(2)$ matrix is given by

$$\begin{pmatrix} a & b \\ \frac{-1}{\sqrt{q}}\,b^* & a^* \end{pmatrix}.$$

The truncation of this algebra by omitting b, b^* and taking the relation obtained from (1) and (2) by eliminating $b\,b^*$ gives

$$a\,a^* - q\,a^*a = 1 - q \tag{6}$$

$$a^*a \leq 1. \tag{7}$$

This algebra, up to a normalization of a, is the q-oscillator algebra and its nontrivial representation (Arik and Coon, 1976) is given by

$$a\,|n\rangle = \sqrt{1-q^n}\,|n-1\rangle \tag{8}$$

$$a^*|n\rangle = \sqrt{1-q^{n+1}}\,|n+1\rangle \tag{9}$$

where $n = 0, 1, 2, \cdots$, such that

$$a\,a^*|n\rangle = (1-q^{n+1})\,|n\rangle. \tag{10}$$

The ket vector $|n\rangle$ describes the quantum state labeled with quantum number n, a^* is the creation operator and a is the annihilation operator. In (10), normalized eigenstates of $a\,a^*$ are labeled by the nonnegative integer n which is related to the occupation number of the q-oscillator. If we instead label the states with the eigenvalue of $a\,a^*$, we obtain

$$a\,a^*\,|\mu\rangle = \mu\,|\mu\rangle \tag{11}$$

$$a\,|\mu\rangle = \sqrt{q^{-1}(\mu+q-1)}\,|q^{-1}(\mu+q-1)\rangle \tag{12}$$

$$a^*|\mu\rangle = \sqrt{\mu}\,|q\,\mu+1-q\rangle \tag{13}$$

where $\mu \equiv 1-q^{n+1}$.

Note that in these expressions q is a label of the algebra whereas μ is related to the representation. A more symmetrical form can be obtained by letting $\nu \equiv 1-q$ so that

$$a\,a^*\,|\mu\rangle = \mu\,|\mu\rangle \tag{14}$$

$$a\,|\mu\rangle = \sqrt{(1-\nu)^{-1}(\mu-\nu)}\,|(1-\nu)^{-1}(\mu-\nu)\rangle \tag{15}$$

$$a^*|\mu\rangle = \sqrt{\mu}\,|\mu+\nu-\mu\,\nu\rangle \tag{16}$$

The action of a^* in (16) resembles the union rule for the membership functions of probabilistic fuzzy sets. If μ is the probability that a given element belongs to fuzzy set A and ν is the probability that the same element belongs to fuzzy set B, then $1-(1-\mu)(1-\nu) = \mu+\nu-\mu\,\nu$ is the probability that it belongs to A $\cup$ B. This relation suggests that an operator algebra related to union and intersection of probabilistic fuzzy sets will have some similarities to the $\mathrm{SU}_q(2)$ algebra given by (1)–(5).

The most important question is the physical meaning of the dimensionless parameter q. In most physical applications $1-q \propto h$ where h is the Planck's constant. Independent of these models but motivated by nonextensive statistical mechanics (Erzan, 1997; Tsallis, 1994) and using the framework of random sets an identification $1-q = \frac{1}{M}$ was made (Arik *et al.*, 1997), where M is the number of elements in the universal set. It was shown that random sets obtained by choosing a random element from the universal set are created by a q-oscillator creation operator and a union of n such random sets has $SU_q(n)$ quantum group symmetry (Arik *et al.*, 1997). When such random sets are transformed into fuzzy sets by taking the one point trace as explained below, the parameter q is fixed and it is related to the membership function μ of an element of the fuzzy set by $\mu = 1 - q = \frac{1}{M}$. In this paper, we will still make the identification $\mu = 1 - q$ but the parameter q will not be fixed. We will show that a consistent algebra possessing operators labeled by all values of $0 < q \leq 1$ can be constructed, leading to the operator algebra of quantized fuzzy sets having a unique representation on the Hilbert space of fuzzy sets.

2. RELATIONS BETWEEN RANDOM SETS, *Q*-OSCILLATORS AND FUZZY SETS

In order to demonstrate the relations between random sets, q-oscillators and fuzzy sets, let us start with an example of a simple universal set $\{e_1, e_2\}$ from which we construct the most general random set

$$\mathrm{P} = \{\{\phi\}_{p_{00}}, \{e_1\}_{p_{10}}, \{e_2\}_{p_{01}}, \{e_1, e_2\}_{p_{11}}\} \tag{17}$$

and say that the set P, for e.g., is equal to $\{e_1\}$ with probability p_{10}. Note that, a classical set is obtained when one of the p_{ij} is set to unity and the others are set to zero. The probabilistic interpretation of the numbers $0 \leq p_{ij} \leq 1$ requires $\sum p_{ij} = 1$. The average number of elements in P is calculated as $p_{00} \times 0 + p_{10} \times 1 + p_{01} \times 1 + p_{11} \times 2$. Let us consider another random set

$$\mathrm{Q} = \{\{\phi\}_{q_{00}}, \{e_1\}_{q_{10}}, \{e_2\}_{q_{01}}, \{e_1, e_2\}_{q_{11}}\} \tag{18}$$

and the union $\mathrm{P} \cup \mathrm{Q}$

$$\mathrm{R} = \mathrm{P} \cup \mathrm{Q} = \{\{\phi\}_{r_{00}}, \{e_1\}_{r_{10}}, \{e_2\}_{r_{01}}, \{e_1, e_2\}_{r_{11}}\} \tag{19}$$

where the probabilities r_{ij} are obtained by multiplying p_{ij} and q_{ij} and summing over all possibilities. For example, $\{e_1\}$ in $\mathrm{P} \cup \mathrm{Q}$ can be obtained by $\{\phi\} \cup \{e_1\}$, $\{e_1\} \cup \{\phi\}$, and $\{e_1\} \cup \{e_1\}$ with respective probabilities $p_{00}q_{00}$, $p_{10}q_{00}$, $p_{10}q_{10}$. Hence, $r_{10} = p_{00}q_{10} + p_{10}q_{00} + p_{10}q_{10}$. Similarly, the intersection S

$$\mathrm{S} = \mathrm{P} \cap \mathrm{Q} = \{\{\phi\}_{s_{00}}, \{e_1\}_{s_{10}}, \{e_2\}_{s_{01}}, \{e_1, e_2\}_{s_{11}}\} \tag{20}$$

is defined with the probabilities $s_{00} = p_{00}(q_{00} + q_{10} + q_{01} + q_{11}) + p_{10}(q_{00} + q_{01}) + p_{01}(q_{00} + q_{10}) + p_{11}q_{00}$, $s_{10} = p_{10}q_{10} + p_{10}q_{11} + p_{11}q_{10}$, $s_{01} = p_{01}q_{01} + p_{01}q_{11} + p_{11}q_{01}$, and $s_{11} = p_{11}q_{11}$.

The relationship of this formulation to that of reference (Hestir *et al.*, 1991; Shafer, 1976) is that given a universal set with M elements we can construct a particular set

$$\mathrm{A} = \left\{\{e_1\}_{\frac{1}{M}}, \{e_2\}_{\frac{1}{M}}, \ldots, \{e_M\}_{\frac{1}{M}}\right\} \tag{21}$$

which can be considered to be a one element random set which is formed by randomly choosing an element from the universal set. It can then be shown that taking the union of A with itself n times gives rise to a random set with an average number of elements $(1-q)^{-1}(1-q^n)$ where $1-q=\frac{1}{M}$ (Arik *et al.*, 1997). The random set obtained in this case is a very special example. However, it can be used to establish the creation operator a^* of the q-oscillator defined by $a\,a^* - q\,a^*a = 1$ as the union operator associated with the random set A (Arik *et al.*, 1997). Thus, as far as the operator algebra generated by a, a^* is concerned, only the average number of elements in a random set is important. The above construction can be made more precise by considering the many-to-one mapping from random sets to probabilistic fuzzy sets. Hence, to any random set P there corresponds a fuzzy set F

$$\mathrm{F} = \{e_1|_{\mu_1}, e_2|_{\mu_2}, \ldots, e_M|_{\mu_M}\} \tag{22}$$

with $0 \le \mu_i \le 1$. The membership function of an element in the fuzzy set is the sum of the probabilities of the subsets of the random set, which contain this element. For $M=2$, we set

$$\begin{aligned}\mu_1 &= p_{10} + p_{11}\\ \mu_2 &= p_{01} + p_{11}\end{aligned} \tag{23}$$

and for $M=3$, we set

$$\begin{aligned}\mu_1 &= p_{100} + p_{110} + p_{101} + p_{111}\\ \mu_2 &= p_{010} + p_{110} + p_{011} + p_{111}\\ \mu_3 &= p_{001} + p_{101} + p_{011} + p_{111}\end{aligned} \tag{24}$$

etc. This rule of obtaining a fuzzy set from a given random set is related to the Dempster-Shafer theory of evidence (Hestir *et al.*, 1991; Shafer, 1976) and is called "taking the one-point trace" (Goodman, 1994; Joslyn, 1996) or "calculating the falling shadow" (Li and Yen, 1995). This gives a specific rule for taking union and intersection of probabilistic fuzzy sets. Fuzzy sets with arbitrary membership functions (Novak, 1989) are possible and one can ask whether an operator algebra describing union and intersection of such sets can be constructed. Such a construction may be expected to yield a relationship among different deformation parameters q. Indeed, we will construct the algebra associated with the operators which we name as union and intersection operators. One important property is that the membership operator $\hat{\mu}$ has continuous representations whereas for the q-oscillator (Arik *et al.*, 1997) it has only discrete representations. Another property is that the operators belonging to different values of q are unified in a single algebra.

3. HILBERT SPACE OF FUZZY SETS

The fuzzy sets we will consider below are obtained from random sets by taking the one-point trace. The membership function μ thus has a probabilistic interpretation and they are probabilistic fuzzy sets. Let us define two fuzzy sets X and Y

$$\begin{aligned} \mathrm{X} &= \{e_1|_{\mu_x(e_1)}, e_2|_{\mu_x(e_2)}, \ldots, e_M|_{\mu_x(e_M)}\} \\ \mathrm{Y} &= \{e_1|_{\mu_y(e_1)}, e_2|_{\mu_y(e_2)}, \ldots, e_M|_{\mu_y(e_M)}\} \end{aligned} \tag{25}$$

where $0 \leq \mu_x(e_i) \leq 1$ and $0 \leq \mu_y(e_i) \leq 1$. The rules of union and intersection for these fuzzy sets are given by

$$\begin{aligned} \mu_{x \cup y}(e_i) &= \mu_x(e_i) + \mu_y(e_i) - \mu_x(e_i)\mu_y(e_i) \\ \mu_{x \cap y}(e_i) &= \mu_x(e_i)\mu_y(e_i). \end{aligned} \tag{26}$$

These rules are usually called (Zadeh, 1965, 1978, Novak, 1989; Zimmermann, 1984) the algebraic sum (probabilistic union) and the algebraic product (probabilistic intersection). They are different from the commonly used union and intersection rules defined in terms of maximum and minimum functions respectively.

We now define a Hilbert space spanned by vectors labeled with $\mu^i = \mu(e_i)$, i.e., the value of the membership function μ for the element e_i. We use Dirac's bra-ket notation (Dirac, 1958) such that vectors are denoted by kets

$$|\,\mu^1, \mu^2, \ldots, \mu^n\rangle \tag{27}$$

whereas dual vectors are denoted by bras

$$\langle\mu^1, \mu^2, \ldots, \mu^n|. \tag{28}$$

We define a Hermitian membership operator $\hat{\mu}^i$ with eigenvalues μ^i

$$\hat{\mu}^i\,|\,\mu^1, \mu^2, \ldots, \mu^n\rangle = \mu^i\,|\,\mu^1, \mu^2, \ldots, \mu^n\rangle. \tag{29}$$

The orthonormality condition is given by

$$\langle\mu^1, \mu^2, \ldots, \mu^n|\,\nu^1, \nu^2, \ldots, \nu^n\rangle = \delta(\mu^1 - \nu^1)\delta(\mu^2 - \nu^2)\cdots\delta(\mu^n - \nu^n) \tag{30}$$

where δ denotes the Dirac delta function.

In accordance with our simple but probably not realistic assumption, the operators $\hat{\mu}^i$ commute among themselves

$$[\hat{\mu}^i, \hat{\mu}^j] = \hat{\mu}^i\hat{\mu}^j - \hat{\mu}^j\hat{\mu}^i = 0 \tag{31}$$

so that they are simultaneously diagonalizable. The membership functions of each element are independent and we will simplify our notation by considering a single element labeled with i. Thus, we can omit the index i and express (29) and (30) as

$$\hat{\mu} \mid \mu\rangle = \mu \mid \mu\rangle,$$
$$\langle\mu \mid \nu\rangle = \delta(\mu - \nu), \quad 0 \le \mu, \nu \le 1. \tag{32}$$

Note that the orthonormal eigenvectors of the Hermitian operator $\hat{\mu}$ form a complete set

$$\int_0^1 \mid \mu\rangle\langle\mu \mid d\mu = I. \tag{33}$$

4. UNION AND INTERSECTION OPERATORS

Next, we will formulate the action of union and intersection operators on this Hilbert space. Let a_q^* denote the operator which unites a fuzzy set with membership function $1 - q$ to a fuzzy set with membership function μ. In accordance with (13) we have

$$a_q^* \mid \mu\rangle = \sqrt{q\mu} \mid q\mu + 1 - q\rangle \tag{34}$$

where an additional factor of $\sqrt{q}$ is inserted for convenience. Similarly, b_q^* is the operator which intersects a fuzzy set with membership function q with a fuzzy set with membership function μ

$$b_q^* \mid \mu\rangle = \sqrt{q(1 - \mu)} \mid q\mu\rangle. \tag{35}$$

The eigenvalues used to label the vectors thus transform in accordance with (26) where in the first equation $\mu_x = 1 - q$, $\mu_y = \mu$ and in the second equation $\mu_x = q$, $\mu_y = \mu$. The fact that operator a^* is labeled by $q = 1 - \mu$ rather than μ is motivated by the notation used for quantum groups. The factors in front of the vectors on the right-hand side of (34) and (35) have been chosen so as to satisfy two conditions: the action of Hermitian conjugates a_q and b_q of a_q^* and b_q^* act on the vectors continuously and they satisfy

$$a_q \, a_q^* = \hat{\mu}$$
$$b_q \, b_q^* = 1 - \hat{\mu}. \tag{36}$$

Note that the right-hand sides of these equations are independent of q and these equations are valid for all values $0 \le q \le 1$. We will prove (36) by obtaining the action of the operators on the vectors. If we multiply (34) from left with $\langle\nu\mid$, we obtain

$$\langle\nu \mid a_q^* \mid \mu\rangle = \sqrt{q\mu}\, \langle\nu \mid q\mu + 1 - q\rangle = \sqrt{q\mu}\, \delta(\nu - (q\mu + 1 - q)) \,,$$
$$\langle\mu \mid a_q \mid \nu\rangle = \overline{\langle\nu \mid a_q^* \mid \mu\rangle} = \sqrt{q\mu}\;\delta(q\mu + 1 - q - \nu) \tag{37}$$

$$= \sqrt{q^{-1}\mu}\,\delta(\mu - q^{-1}(\nu + q - 1))$$
$$= \sqrt{q^{-2}(\nu + q - 1)}\,\langle \mu\,|q^{-1}(\nu + q - 1)\rangle\theta(\nu + q - 1) \tag{38}$$

where

$$\theta(x) = \begin{Bmatrix} 1\ , & x > 0 \\ 0\ , & x \leq 0 \end{Bmatrix} \tag{39}$$

is the unit step function. The θ function has to be inserted when the δ function is replaced by the inner product. This is because of the fact that the label of the ket vector has to be between zero and one. By multiplying (38) from left with $|\,\mu\rangle$ and integrating over μ, using the completeness relation (33) gives

$$\begin{aligned} a_q|\,\nu\rangle &= \int_0^1 |\,\mu\rangle\langle\mu\,|a_q|\,\nu\rangle\,d\mu \\ &= \int_0^1 |\,\mu\rangle\sqrt{q^{-1}\mu}\,\delta(\mu - q^{-1}(\nu + q - 1))\,d\mu \\ &= \sqrt{q^{-2}(\nu + q - 1)}\,|q^{-1}(\nu + q - 1)\rangle\theta(\nu + q - 1). \end{aligned} \tag{40}$$

Hence, a_q acting on (34) from left yields

$$\begin{aligned} a_q\,a_q^*|\,\mu\rangle = \sqrt{q\mu}\,a_q|\,q\mu + 1 - q\rangle &= \sqrt{q\mu}\,\sqrt{q^{-2}[(q\mu + 1 - q) + q - 1]}\,|q^{-1} \\ &\times\,[(q\mu + 1 - q) + q - 1]\rangle\theta(q\mu) = \mu\,|\mu\rangle = \hat{\mu}\,|\mu\rangle. \end{aligned} \tag{41}$$

Similarly, from (35) one obtains

$$\langle\nu\,|b_q^*|\,\mu\rangle = \sqrt{q(1-\mu)}\,\langle\nu\,|\,q\mu\rangle = \sqrt{q(1-\mu)}\,\delta(\nu - q\mu) \tag{42}$$

$$\begin{aligned} \langle\mu\,|b_q|\,\nu\rangle &= \overline{\langle\nu\,|b_q^*|\,\mu\rangle} = \sqrt{q(1-\mu)}\,\delta(q\mu - \nu) \\ &= \sqrt{q^{-1}(1-\mu)}\,\delta(\mu - q^{-1}\nu) \\ &= \sqrt{q^{-1}(1 - q^{-1}\nu)}\,\langle\mu\,|\,q^{-1}\nu\rangle\,\theta(q - \nu). \end{aligned} \tag{43}$$

By multiplying (43) from left with $|\,\mu\rangle$ and integrating over μ, using the completeness relation (33) gives

$$\begin{aligned} b_q|\,\nu\rangle &= \int_0^1 |\,\mu\rangle\langle\mu\,|b_q|\,\nu\rangle\,d\mu \\ &= \int_0^1 |\,\mu\rangle\sqrt{q^{-1}(1-\mu)}\,\delta(\mu - q^{-1}\nu)\,d\mu \\ &= \sqrt{q^{-2}(q - \nu)}\,|\,q^{-1}\nu\rangle\,\theta(q - \nu). \end{aligned} \tag{44}$$

Hence, b_q acting on (35) from left yields

$$\begin{aligned} b_q\, b_q^* \,|\, \mu\rangle &= \sqrt{q(1-\mu)}\, b_q \,|\, q\mu\rangle \\ &\sqrt{q(1-\mu)} = \sqrt{q^{-1}(1-\mu)}\,|\,\mu\rangle\, \theta(q - q\mu) \\ &= (1-\mu)|\,\mu\rangle = (1-\hat{\mu})|\,\mu\rangle. \end{aligned} \tag{45}$$

We have shown that the union operator a_q^*, the intersection operator b_q^* and their Hermitian conjugates a_q, b_q act on the Hilbert space of fuzzy sets spanned by the normalized vectors $|\,\mu\rangle$ as

$$\begin{aligned} a_q|\,\mu\rangle &= \sqrt{q^{-2}(\mu+q-1)}\,|q^{-1}(\mu+q-1)\rangle\theta(\mu+q-1) \\ a_q^*|\,\mu\rangle &= \sqrt{q\mu}\,|\,q\mu+1-q\rangle \\ b_q|\,\mu\rangle &= \sqrt{q^{-2}(q-\mu)}\,|\,q^{-1}\mu\rangle\,\theta(q-\mu) \\ b_q^*|\,\mu\rangle &= \sqrt{q(1-\mu)}\,|\,q\mu\rangle \end{aligned} \tag{46}$$

where $0 < q \leq 1$ and $0 \leq \mu \leq 1$. It can also be shown that these operators satisfy the following algebraic relations

$$\begin{aligned} a_1 &= a_1^*, b_1 = b_1^*, a_1^* a_1 + b_1^* b_1 = 1 \\ a_q a_q^* &= a_1^* a_1, b_q b_q^* = b_1^* b_1 \\ a_q b_1 &= \sqrt{q} b_1 a_q, a_q b_1^* = \sqrt{q}\, b_1^* a_q \\ b_q a_1 &= \sqrt{q}\, a_1 b_q, b_q a_1^* = \sqrt{q}\, a_1^* b_q \end{aligned} \tag{47}$$

and

$$a_q a_q^* = (a_1)^2, b_q b_q^* = (b_1)^2, a_1^2 + b_1^2 = 1. \tag{48}$$

Using the following relations

$$\begin{aligned} a_{q_1}^* a_1 a_{q_2}^* a_1 \,|\,\mu\rangle &= \sqrt{q_1 q_2}\,(q_2\mu^2 + (1-q_2)\mu)\,|\,q_1 q_2 \mu + 1 - q_1 q_2\rangle \\ a_{q_2}^* a_1 a_{q_1}^* a_1 \,|\,\mu\rangle &= \sqrt{q_1 q_2}\,(q_1\mu^2 + (1-q_1)\mu)\,|\,q_1 q_2 \mu + 1 - q_1 q_2\rangle \\ a_{q_1 q_2}^* a_1 \,|\,\mu\rangle &= \sqrt{q_1 q_2}\,\mu\,|\,q_1 q_2 \mu + 1 - q_1 q_2\rangle \end{aligned}$$

one obtains

$$q_1 a_{q_1}^* a_1 a_{q_2}^* - q_2 a_{q_2}^* a_1 a_{q_1}^* = (q_1 - q_2)\, a_{q_1 q_2}^*. \tag{49}$$

The Hermitian conjugate of this relation is given by

$$q_1 a_{q_1} a_1^* a_{q_2} - q_2 a_{q_2} a_1^* a_{q_2} = (q_1 - q_2)\, a_{q_1 q_2}. \tag{50}$$

Similar relations are also valid for b and b^*

$$q_1 b^*_{q_1} b_1 b^*_{q_2} - q_2 b^*_{q_2} b_1 b^*_{q_1} = (q_1 - q_2)\, b^*_{q_1 q_2} \tag{51}$$

$$q_1 b_{q_1} b^*_1 b_{q_2} - q_2 b_{q_2} b^*_1 b_{q_2} = (q_1 - q_2)\, b_{q_1 q_2}. \tag{52}$$

The relations below

$$a^*_1 a_{q_1} a^*_{q_2} a_1 \mid \mu\rangle = \sqrt{\frac{q_2}{q_1}} \left[\frac{q_2}{q_1}\mu^2 + (1 - \frac{q_2}{q_1})\mu\right] \left| \frac{q_2}{q_1}\mu + 1 - \frac{q_2}{q_1}\right\rangle \theta\left(\mu - 1 + \frac{q_2}{q_1}\right)$$

$$a^*_{q_2/q_1} a_1 a^*_1 a_1 \mid \mu\rangle = \sqrt{\frac{q_2}{q_1}}\, \mu^2 \left| \frac{q_2}{q_1}\mu + 1 - \frac{q_2}{q_1}\right\rangle$$

$$a^*_{q_2/q_1} a_1 \mid \mu\rangle = \sqrt{\frac{q_2}{q_1}}\, \mu \left| \frac{q_2}{q_1}\mu + 1 - \frac{q_2}{q_1}\right\rangle$$

lead to

$$a_{q_1} a^*_{q_2} = a_1 a^*_{q_2/q_1}, \quad q_2 < q_1 \tag{53}$$

$$a_{q_1} a^*_{q_2} = a_{q_1/q_2} a^*_1, \quad q_1 < q_2. \tag{54}$$

By taking $q_1/q_2 = q_3/q_4$ and using (54) one obtains

$$a_{q_1} a^*_{q_2} = a_{q_3} a^*_{q_4}. \tag{55}$$

In addition, we can use the following relations

$$a_{q_1} b^*_{q_2} \mid \mu\rangle = q_1^{-1}\sqrt{q_2(1-\mu)}\sqrt{q_2\mu + q_1 - 1}\mid q_1^{-1}(q_2\mu + q_1 - 1)\rangle \times \theta(q_2\mu + q_1 - 1)$$

$$b^*_{q_3} a_{q_4} \mid \mu\rangle = q_4^{-1}\sqrt{q_3(1-\mu)}\sqrt{q_4^{-1}(\mu + q_4 - 1)}\mid q_4^{-1} q_3(\mu + q_4 - 1)\rangle \times \theta(\mu + q_4 - 1)$$

to obtain

$$a_{q_1} b^*_{q_2} = \sqrt{q_3 q_4}\, b^*_{q_3} a_{q_4} \tag{56}$$

where $1 - q_1 = q_2(1 - q_4)$ and $q_2/q_1 = q_3/q_4$. Finally, the relations

$$a^*_{q_1} b^*_{q_2} \mid \mu\rangle = q_2\sqrt{q_1}\sqrt{\mu(1-\mu)}\mid q_1 q_2 \mu + 1 - q_1\rangle$$

$$b^*_{q_3} a^*_{q_4} \mid \mu\rangle = q_4\sqrt{q_3}\sqrt{\mu(1-\mu)}\mid q_3 q_4 \mu + q_3(1 - q_4)\rangle$$

give us

$$\sqrt{q_1}\, a^*_{q_1} b^*_{q_2} = \sqrt{q_3}\, b^*_{q_3} a^*_{q_4} \tag{57}$$

or

$$\sqrt{q_4}\, a^*_{q_1} b^*_{q_2} = \sqrt{q_2}\, b^*_{q_3} a^*_{q_4} \tag{58}$$

where q_1, q_2, q_3, q_4 are not independent but satisfy $1 - q_1 = q_3(1 - q_4)$ and $q_1 q_2 = q_3 q_4$. We can also express these as follows

$$q_3 = 1 - q_1 + q_1 q_2, \quad q_4 = q_1 q_2 / q_3. \tag{59}$$

5. OPERATOR ALGEBRA OF QUANTIZED FUZZY SETS

Now, we would like to address the question of what (minimal) subset of the algebraic relations computed above is sufficient to prove that the representation is given by (46). We will prove that the relations (49), (51) together with some supplementary relations given in (47) uniquely specify the algebra. Hence, we start with the following relations

$$q_1 a^*_{q_1} a_1 a^*_{q_2} - q_2 a^*_{q_2} a_1 a^*_{q_1} = (q_1 - q_2)\, a^*_{q_1 q_2} \tag{60}$$

$$q_1 b^*_{q_1} b_1 b^*_{q_2} - q_2 b^*_{q_2} b_1 b^*_{q_1} = (q_1 - q_2)\, b^*_{q_1 q_2} \tag{61}$$

$$a_1 = a^*_1, \quad b_1 = b^*_1 \tag{62}$$

$$a^*_1 a_1 + b^*_1 b_1 = 1 \tag{63}$$

for all $0 < q_1, q_2 \leq 1$. These relations are invariant under the interchange of a_q and b_q. For simplicity we will define

$$A_q \equiv a^*_1 a_q, \quad B_q \equiv b^*_1 b_q \tag{64}$$

so that (60) and (61) reduces to

$$q_1 A_{q_2} A_{q_1} - q_2 A_{q_1} A_{q_2} = (q_1 - q_2)\, A_{q_1 q_2} \tag{65}$$

$$q_1 B_{q_2} B_{q_1} - q_2 B_{q_1} B_{q_2} = (q_1 - q_2)\, B_{q_1 q_2}. \tag{66}$$

We denote the eigenvalue of A_1, A^*_1 by μ, then from (63) the eigenvalue of B_1, B^*_1 becomes $1 - \mu$:

$$A_1 |\, \mu\rangle = A^*_1 |\, \mu\rangle = \mu\, |\, \mu\rangle \tag{67}$$

$$B_1 |\, \mu\rangle = B^*_1 |\, \mu\rangle = (1 - \mu)\, |\, \mu\rangle. \tag{68}$$

Since A_1 and B_1 must be nonnegative operators, we must have $0 \leq \mu \leq 1$.

Using these relations, we will derive the actions of A_q, A^*_q, B_q, B^*_q on vectors $|\, \mu\rangle$. In (65) taking $q_1 = 1$ and $q_2 = q$ gives

$$A_q A_1 = q A_1 A_q + (1 - q) A_q.$$

By taking the Hermitian conjugate of both sides, one can obtain the action of the operators A_q^* on vectors $|\,\mu\rangle$

$$A_1^* A_q^* |\,\mu\rangle = q A_q^* A_1 |\,\mu\rangle + (1-q) A_q^* |\,\mu\rangle = (q\mu + 1 - q) A_q^* |\,\mu\rangle.$$

It follows that A_q^* must be of the form

$$A_q^* \,|\,\mu\rangle = C(\mu)\,|q\mu + 1 - q\rangle. \tag{69}$$

Hence, using the orthonormality condition $\langle \mu \,|\, \nu\rangle = \delta(\mu - \nu)$, we obtain

$$C(\mu) = \sqrt{q}\,\mu.$$

By substituting $C(\mu)$ in (69) one obtains

$$A_q^* \,|\,\mu\rangle = \sqrt{q}\,\mu\,|q\mu + 1 - q\rangle. \tag{70}$$

The calculation of $A_q \,|\,\mu\rangle$ is slightly more tricky. Multiplying (70) from left by $\langle \nu\,|$ and taking complex conjugate of both sides and using the orthonormality condition, we find

$$\begin{aligned} \langle \mu\,|A_q\,|\,\nu\rangle &= \sqrt{q}\,\mu\,\delta(q\mu + 1 - q - \nu) = \sqrt{q^{-1}}\,\mu\,\delta(\mu + q^{-1} - 1 - q^{-1}\nu) \\ &= \sqrt{q^{-1}}\,\mu\,\langle \mu\,|\,q^{-1}(\nu + q - 1)\rangle\,\theta(q^{-1}(\nu + q - 1)) \end{aligned} \tag{71}$$

so that

$$A_q\,|\,\nu\rangle = \sqrt{q^{-1}}\,[\,q^{-1}(\nu + q - 1)]\,|\,q^{-1}(\nu + q - 1)\rangle\,\theta(\nu + q - 1). \tag{72}$$

Similarly, in (66) taking $q_2 = 1$ and $q_1 = q$ gives

$$B_q B_1 = q B_1 B_q + (1-q) B_q.$$

After taking the Hermitian conjugate of both sides one can obtain the action of B_q^* operators on vectors $|\,\mu\rangle$

$$B_1^* B_q^* |\,\mu\rangle = q B_q^* B_1 |\,\mu\rangle + (1-q) B_q^* |\,\mu\rangle = (1 - q\mu) B_q^* |\,\mu\rangle.$$

Hence B_q^* must be of the form

$$B_q^*\,|\,\mu\rangle = K(\mu)\,|q\mu\rangle. \tag{73}$$

Using the condition $\langle \mu\,|\,\nu\rangle = \delta(\mu - \nu)$, we get

$$K(\mu) = \sqrt{q}\,(1 - \mu).$$

By substituting $K(\mu)$ in (73) one obtains

$$B_q^*\,|\,\mu\rangle = \sqrt{q}\,(1 - \mu)\,|q\mu\rangle. \tag{74}$$

Multiplying (74) from left by $\langle \nu\,|$ and taking complex conjugate of both sides and using the orthonormality condition, we obtain

$$\begin{aligned} \langle \mu\,|B_q\,|\,\nu\rangle &= \sqrt{q}\,(1 - \mu)\,\delta(q\mu - \nu) = \sqrt{q^{-1}}\,(1 - \mu)\,\delta(\mu - q^{-1}\nu) \\ &= \sqrt{q^{-1}}\,(1 - \mu)\,\langle \mu\,|\,q^{-1}\nu\rangle\,\theta(q^{-1}(q - \nu)) \end{aligned} \tag{75}$$

which yields

$$B_q\,|\,\nu\rangle = \sqrt{q^{-1}}\,(1 - q^{-1}\nu)\,|\,q^{-1}\nu\rangle\,\theta(q - \nu). \tag{76}$$

Let us now consider the Hermitian operator $a_1 = a_1^*$ and its eigenvector $|\,\alpha\rangle$ with eigenvalue α

$$a_1\,|\,\alpha\rangle = \alpha\,|\,\alpha\rangle, \quad \alpha \in R. \tag{77}$$

Then A_1 has eigenvalue α^2

$$A_1\,|\,\alpha\rangle = a_1^2\,|\,\alpha\rangle = \alpha^2\,|\,\alpha\rangle.$$

Hence we can set

$$|\,\alpha\rangle = |\,\mu, \operatorname{sgn}\alpha\rangle, \quad \alpha = \pm\sqrt{\mu}.$$

The two signs of α correspond to $a_q \to -a_q$ symmetry of the algebra (60)–(63). The vectors corresponding to different values of the sign of α are orthogonal and we have the following relations

$$a_1\,|\,\mu, +\rangle = \sqrt{\mu}\,|\,\mu, +\rangle$$
$$a_1\,|\,\mu, -\rangle = -\sqrt{\mu}\,|\,\mu, -\rangle$$
$$\langle\mu, +\,|\,\nu, -\rangle = 0.$$

Thus, the vectors with only one value of the sign form an irreducible representation. Henceforth, we will only consider the plus sign since the other sign can also follow a similar treatment. From (64), (70) and (77) we have

$$A_q^*\,|\,\mu\rangle = a_q^* a_1|\,\mu\rangle = \sqrt{\mu}\,a_q^*|\,\mu\rangle = \sqrt{q}\,\mu\,|\,q\mu + 1 - q\rangle$$

so that

$$a_q^*|\,\mu\rangle = \sqrt{q\mu}\,|\,q\mu + 1 - q\rangle. \tag{78}$$

Multiplying (78) from left with the bra vector $\langle\nu\,|$, taking complex conjugation of both sides and using the orthonormality condition we find

$$\langle\mu\,|\,a_q\,|\,\nu\rangle = \sqrt{q\mu}\,\delta(q\mu + 1 - q - \nu) = \sqrt{q^{-1}\mu}\,\delta(\mu - q^{-1}(\nu + q - 1))$$
$$= \sqrt{q^{-1}\mu}\,\langle\mu\,|\,q^{-1}(\nu + q - 1)\rangle\,\theta(q^{-1}(\nu + q - 1))$$

which yields

$$a_q|\,\nu\rangle = \sqrt{q^{-2}(\nu + q - 1)}\,|\,q^{-1}(\nu + q - 1)\rangle\,\theta(\nu + q - 1). \tag{79}$$

In a similar manner, when we consider the Hermitian operator $b_1 = b_1^*$, (68) shows that we can set

$$b_1|\,\mu\rangle = \sqrt{1 - \mu}\,|\,\mu\rangle.$$

From (64), (74) and (80) we can write

$$B_q^* \,|\, \mu\rangle = b_q^* b_1 |\, \mu\rangle = \sqrt{1-\mu}\, b_q^* \,|\, \mu\rangle = \sqrt{q}\,(1-\mu)\,|\, q\mu\rangle$$

and obtain

$$b_q^* |\, \mu\rangle = \sqrt{q(1-\mu)}\,|\, q\mu\rangle. \tag{80}$$

To calculate $b_q \mid \nu\rangle$ we follow the same steps as before

$$\begin{aligned}\langle \mu \mid b_q \mid \nu\rangle &= \sqrt{q(1-\mu)}\,\delta(q\mu - \nu) = \sqrt{q^{-1}(1-\mu)}\,\delta(\mu - q^{-1}\nu)\\ &= \sqrt{q^{-1}(1-\mu)}\,\langle \mu \,|\, q^{-1}\nu\rangle\,\theta(q^{-1}(q-\nu))\end{aligned}$$

so that

$$b_q \mid \nu\rangle = \sqrt{q^{-2}(q-\nu)}\,|\, q^{-1}\nu\rangle\,\theta(q-\nu). \tag{81}$$

The relations (78)–(81) are the same as the relations given in (46). Hence, the algebra of quantized fuzzy sets given by (60)–(63) has the quantized fuzzy set representation.

6. CONCLUSION

In conclusion, we can state that we were able to construct a consistent operator algebra of quantized fuzzy sets starting from the relations (34) and (35) where a_q^* and b_q^* are the operator versions of union and intersection rules for probabilistic fuzzy sets, respectively. The relations (47) and (48) satisfied by these operators bear a striking resemblance to the operator algebra (1)–(5) satisfied by the elements of quantum group matrix $SU_q(2)$. In fact, if we pick only the relations involving a_q, a_q^*, b_1, b_1^* in (47) and (48) we obtain

$$a_q\, a_q^* + b_1\, b_1^* = 1 \tag{82}$$

$$b_1\, b_1^* = b_1^*\, b_1 \tag{83}$$

$$a_q\, b_1 = \sqrt{q}\, b_1\, a_q \tag{84}$$

$$a_q\, b_1^* = \sqrt{q}\, b_1^*\, a_q. \tag{85}$$

There exist two differences between (1)–(5) and (82)–(85). One is that the first equality (1) is missing, the second is that in (82)–(85) $b_1 = b_1^*$ whereas in (1)–(5) there is no such requirement. For the algebra (1)–(5) imposing the condition $b = b^*$ yields a consistent algebra, however this will destroy the quantum group related co-algebra structure of $SU_q(2)$.

We hope that further investigations along the lines of this work will yield an understanding of the relationship between fuzzy sets and quantum physics. In this respect a consistent algebra where membership functions of different elements do not commute will be desirable.

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